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SOME FINITELY PRESENTED GROUPS OF COHOMOLOGICAL DIMENSION TWO WITH PROPERTY (FA)

Stephen J. PRIDE
Department of Mathematics, University Gardens, Glasgow G12 8QW, Scotland

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In Problem C3 of [6], it is asked whether every finitely generated group of cohomological dimension two splits over a free group. In [1] it is remarked that the answer to this problem is 'no' in general: a group S constructed by Ol'sanskii in [4] provides an example showing this. At the Geometric Topology Conference, University of Sussex, 1982, M.J. Dunwoody asked about the status of the above problem for finitely presented groups (S is not finitely presented). It is shown here that the answer is 'no' in this case also.

In fact, a stronger result is obtained. Recall [5] that a group G is said to have property (FA) if, whenever G acts (without inversions) on a tree there is at least one fixed point. For finitely generated groups this is equivalent [5, Chapter I, Theorem 15] to the following two conditions: G has no infinite cyclic quotient; G is not an amalgam. I give here examples of finitely presented groups of cohomological dimension two which have property (FA). It should be remarked that the group S above has property (FA).

Let G be generated by two elements x, y subject to six defining relators of the following form:

$$R_1(x,y) = xy^{\alpha_1}xy^{\alpha_2}\cdots xy^{\alpha_k},$$

$$R_2(x,y) = yx^{\beta_1}yx^{\beta_2}\cdots yx^{\beta_k},$$

$$R_3(x,y) = x^{\gamma_1}y^{-\delta_1}x^{\gamma_2}y^{-\delta_2}\cdots x^{\gamma_k}y^{-\delta_k},$$

$$R_4(x,y) = xy^{-p_1}xy^{p_1}xy^{-p_2}xy^{p_2}\cdots xy^{-p_k}xy^{p_k},$$

$$R_5(x,y) = yx^{-q_1}yx^{q_1}yx^{-q_2}yx^{q_2}\cdots yx^{-q_k}yx^{q_k},$$

$$R_6(x,y) = (xy)^{m_1}(x^{-1}y^{-1})^{n_1}(xy)^{m_2}(x^{-1}y^{-1})^{n_2}\cdots(xy)^{m_k}(x^{-1}y^{-1})^{n_k}.$$

Here all the integers $\alpha_i, \beta_i, \gamma_i, \delta_i, p_i, q_i, m_i, n_i$ are positive. It will be shown shortly that G has property (FA).

Now if p is a positive integer, then for a suitable choice of k and the α_i, β_i etc., the symmetrized closure of $\{R_1, R_2, \ldots, R_6\}$ can be made to satisfy the small cancellation condition $C(p)$. Moreover, one can arrange that no R_j is a proper power. Then if $p \geq 6$, it follows from [1-3] (and the fact that G is not free) that the cohomological dimension of G is two.

To see that G has property (FA), first note that G has no infinite cyclic quotient, for in the abelianization of G, x and y have finite order (by R_4 and R_5).

Secondly, G is not an amalgam. For suppose $G = A_{-1} *_C A_1$ with $C \neq A_{\pm 1}$. Each element $g \in G$ can be writen in normal form $t_1 t_2 \cdots t_r$ where the t_i come alternately from the factors $A_{\pm 1}$, and no t_i belongs to C unless $r=1$. We write $|g| = r$.

Now we may assume without loss of generality that no pair (u, v) conjugate to (x, y) has $|u| + |v| < |x| + |y|$. Then taking a suitable conjugate of (x, y) and interchanging x and y, if necessary, we may write

$$x = x_1 x_2 \cdots x_m, \qquad y = z_l^{-1} \cdots z_1^{-1} y_1 \cdots y_n z_1 \cdots z_l \quad (l \geq 0,\ n > 0)$$

in normal form. Here: $m \leq n$; if $m > 1$, then $x_m x_1 \notin C$ (that is, x is cyclically reduced); if $n > 1$, then $y_n y_1 \notin C$; if $l \neq 0$, then x_1, x_m belong to the same factor and z_l does not belong to this factor.

By considering the normal form of $R_6(x, y)$ one easily deduces that $l = 0$. Moreover, $n > 1$, for if $m = n = 1$, then x_1, y_1 could not belong to the same factor (since $\mathrm{sgp}\{x, y\} = G$), and so $R_6(x, y) \neq 1$.

If $m > 1$, then by considering $R_1(x, y)$ it is easily deduced that one of $x_m y_1$, $y_n x_1$ belongs to C. Thus $x_m y_n^{-1}$, $y_1^{-1} x_1 \notin C$, so $R_3(x, y) \neq 1$, a contradiction. Hence $m = 1$.

Now by minimality, not all of x, y_1, y_n can belong to the same factor. By considering the normal form of $R_1(x, y)$ one deduces that y_1, y_n must belong to different factors, and one of $y_n x, x y_1$ – say the former – belongs to C. But then $y_n x y_n^{-1} \notin C$, so $R_4(x, y) \neq 1$.

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